Reliability of Uncertain Laminated Shells Due to Buckling and Supersonic Flutter

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An objective of this paper is to study the supersonic flutter characteristics of initially compressed, laminated, thin shell structures using a 48-degree-of-freedom, doubly curved, quadrilateral, laminated, thin shell finite element developed on the basis of the Kirchhoff-Love thin shell theory and classical lamination theory. The aerodynamic pressure due to supersonic potential flow is described by the piston theory. Another objective of this paper is to study the reliability of initially compressed, laminated thin shells with structural uncertainties due to variabilities that occurred during fabrication. The two failure criteria considered are buckling and supersonic flutter. Interactive effects between the middle surface load and aerodynamic pressure for the uncertain shells are also studied. For buckling analysis, the structural uncertainties include modulus of elasticity, thickness, and fiber orientation of individual lamina, as well as geometric imperfections of the entire shell. For flutter analysis, additional uncertainties of mass densities of both the shell and the air, as well as the amplitudes of uniformly and linearly distributed middle surface loads, are considered. The stochastic element formulation is defined by including the effects of structural uncertainties and random middle surface loads. The stochastic solution procedure is developed based on the mean-centered, second-moment perturbation technique. To evaluate the validity and to demonstrate the applicability of the present formulation and solution procedure, a series of vibration, buckling, and supersonic flutter analyses of thin shells with structural uncertainties under random middle surface loads are performed. The results quantify the effects of these uncertain parameters on the reduction of the structural reliability and stability boundaries of initially compressed laminated shells. The results also provide physical insight into such design and fabrication problems with practical significance.

Introduction

THIN shells are a popular and useful form of structural components with many significant applications in aerospace and other fields of engineering. For the obvious advantages, such as the high strength-to-weight ratio and high stiffness-to-weight ratio, etc., fiber reinforced laminated composite materials have been increasingly used in the design and fabrication of thin shell structures.

While there are many advantageous and desirable features of thin laminated shells, there also exist the undesirable problems of vibration, buckling, and flutter, etc. Possible variabilities that occur during fabrication further complicate the analysis and design aspects of these problems as well as the structural performance and reliability.

Many investigators studied the vibration and buckling behaviors of laminated thin shell structures. Some recent brief reviews for such studies were given by, among others, Yang et al.[1] and Kapania.[2] For the supersonic flutter analysis of thin wall structures, most studies were devoted to laminated plates[3-6] and isotropic shells.[7-11] Fewer investigators treated the laminated composite shells. For example, Librescu[12] investigated the supersonic flutter behavior of anisotropic plates and shells. Sunder et al.[13] studied the flutter boundaries of a three-ply orthotropic conical shell.

Due to the difficulty frequently encountered in achieving quality control of manufacturing process, structural variabilities may occur that are random in nature. These process-induced variabilities may include fiber size, volume fraction of fibers, fiber orientations, thickness of each lamina, and curvature of the shell. Such variabilities will affect the achievable performance of strengths and stiffnesses of the finished shells and will also influence their structural reliabilities.

Studies of the uncertain structural parameters have been mostly on the effects of imperfections on vibration and buckling of thin shell structures.[14-22] In recent years, the interests in random uncertainties have been extended beyond the random geometric imperfections to include other possible uncertainties such as modulus of elasticity,[23] mass density,[23] thickness,[24] and fiber orientations,[25] etc. It appears that in most past studies, only a single uncertain structural parameter was considered, with random geometric imperfection attracting the most attention. However, it is important and desirable to consider the combined effects of several essential structural uncertainties in the analysis, design, and safety assessment of structures. Such an idea was proposed[26,27] for the probabilistic seismic safety study of an existing nuclear power plant. The authors have developed a stochastic 16-degree-of-freedom

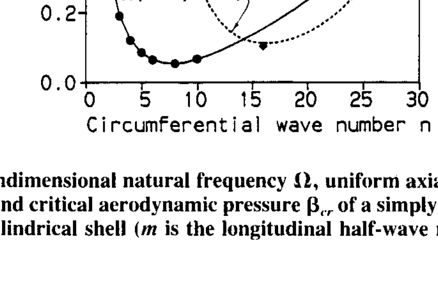

Fig. 1 Nondimensional natural frequency Ω, uniform axial buckling stress N_{cr}, and critical aerodynamic pressure β_{cr} of a simply supported isotropic cylindrical shell (*m* is the longitudinal half-wave number).

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(DOF) quadrilateral laminated thin plate finite element[28] to study the combined effects of several uncertain structural parameters on the reliability and stability boundaries of plate structures.

It is the intent of this study to extend this earlier plate formulation to a 48-DOF doubly curved quadrilateral thin shell finite element to investigate the effects of several random structural uncertainties on the buckling and supersonic flutter behaviors as well as the reliabilities of laminated thin shells.

The shell is assumed to be thin so that the Kirchhoff-Love thin shell theory and classical lamination theory are applicable. The effect of the transverse shear deformations is neglected. The aerodynamic pressure due to supersonic potential flow is described using the two-dimensional quasisteady supersonic theory. The effect of structural and aerodynamic dampings on the determination of the characteristic of the flutter boundary was studied previously by, among others, Librescu and Badoiu.[29] The structural and aerodynamic nonlinearities also have a significant effect on the determination of the flutter characteristic. Such effect has been studied by, among others, Librescu[30,31] and Han and Yang.[32] However, both effects of dampings and nonlinearities are neglected here.

The formulation includes the linear stiffness, incremental stiffness, mass, and aerodynamic matrices. These matrices are random in nature, depending on the probabilistic properties of the uncertainties. A solution procedure within the framework of stochastic finite element method[33,34] is developed based on the mean-centered, second-moment perturbation technique.

The random structural uncertainties considered include modulus of elasticity, mass density, thickness, and fiber orientation of individual lamina, as well as initial geometric imperfections of the entire shell. In the design of thin shell structures, it is often necessary to consider the effect of middle surface stresses. It is apparently interesting to study further the effect of middle surface stress as a random variable on the critical aerodynamic pressure or flutter speed of the laminated thin shells. Such study is another objective of this paper.

To establish the validity of the present stochastic thin shell element formulation and solution procedure, vibration, buckling, and supersonic flutter analyses of a series of examples of isotropic cylindrical and conical shells, orthotropic conical shells, and laminated cylindrical shells are first solved with results compared with existing alternative solutions. To demonstrate the practical applicability of the present method to simultaneously treat a variety of structural uncertainties, a series of buckling, supersonic flutter, and reliability analyses of examples of simply supported thin cylindrical shells with uncertain modulus of elasticity, mass density, thickness, fiber orientations, and random imperfections under random middle surface loads are performed. Some representative results are verified using the Monte-Carlo simulation approach. The results quantify the effects of these uncertain parameters on the reduction of the structural reliability and stability boundaries of initially compressed laminated shells. The results also provide physical insight into such design and fabrication problems with practical interests.

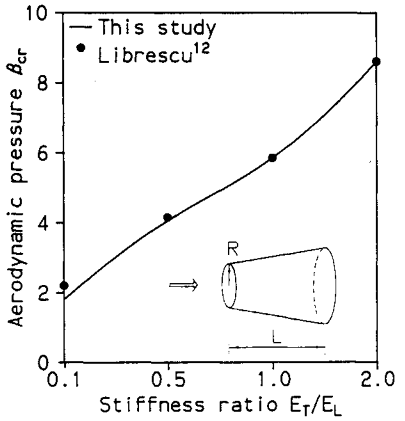

Fig. 2 Supersonic flutter boundary for a simply supported orthotropic conical shell.

Formulation

An earlier formulation developed by the authors for a stochastic plate element for structural reliability analysis is extended here to formulate a stochastic shell element. A brief summary of the salient features of the formulation is given below.

Strain-Displacement Relations

The strain-displacement relations for imperfect shells are represented in terms of Cartesian components that are an extended version of the strain-displacement relations given in tensor form by Niordson.[35] The effects of imperfections are included by modifying the tangential strain-displacement relations. These are given as

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}(f^i_{,\alpha}u^i_{,\beta} + f^i_{,\beta}u^i_{,\alpha} + u^i_{,\alpha}u^i_{,\beta} + u^i_{,\alpha}v^i_{,\beta} + u^i_{,\beta}v^i_{,\alpha}) \quad (1)$$

where f^i, u^i, and v^i are the Cartesian components of the middle surface, displacement, and imperfection at a given point on the shell surface, respectively; i varies from 1 to 3 and α and β from 1 to 2, respectively. In this study, the effects of imperfections on the curvature-displacement relations are ignored.

Laminate Constitutive Relations

The laminated anisotropic construction of the shell is assumed to be made up of n layers. Each lamina is assumed to be orthotropic with its principal material axes at an angle with the local coordinate axes. The stress-strain relation for each layer is expressed with reference to the local coordinate system through coordinate transformation. The stress and moment resultants are then related to the middle surface strain ε and the change of curvature κ as[36]

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} \quad (2)$$

where $\{N\}$ is the vector of resultant tangential forces, $\{M\}$ is the vector of resultant moments, and the coefficients in matrices $[A]$, $[B]$, and $[D]$ are given as

$$[A_{ij}, B_{ij}, D_{ij}] = \int_{-h/2}^{h/2} Q_{ij}\,(1,z,z^2)\,\mathrm{d}z \qquad (i,j = 1,2,6) \quad (3)$$

with Q_{ij} denoting the plane-stress stiffness for individual layer and h the total shell thickness of the shell.

Aerodynamic Pressure

The aerodynamic pressure due to supersonic potential flow is described by the two-dimensional quasisteady supersonic theory as

$$p(x,y,t) = \frac{2q}{\sqrt{M_\infty^2 - 1}}\left(\frac{\partial w}{\partial \zeta} + \frac{1}{V}\frac{M_\infty^2 - 2}{M_\infty^2 - 1}\frac{\partial w}{\partial t}\right) \quad (4)$$

where q is the dynamic pressure, M_∞ the Mach number, V the flow velocity, and ζ the direction of the flow in the global coordinate system.

By following the procedure proposed by Sander et al.[37] and Yang,[38] the explicit form for the coefficient in the aerodynamic matrix can be obtained as the following general form:

$$a_{ij} = \bar{\beta}\int_A N_i \frac{\partial N_j}{\partial \eta}\frac{\partial \eta}{\partial \zeta}\,\mathrm{d}A \quad (5)$$

where N_i is the shape function for the ith degree of freedom, η is the direction of the flow in the local element coordinate, and β is defined as

$$\bar{\beta} = \frac{2q}{\sqrt{M_\infty^2 - 1}} \tag{6}$$

Random Structural Uncertainties

The random field of a single structural uncertainty is described as a random amplitude multiplied by a deterministic spatial function. The random amplitude ξ_i can be expressed as the sum of its mean value $E[\xi_i]$ and a random variable α_i as

$$\xi_i = E[\xi_i] + \alpha_i \tag{7}$$

The mean value of random variable α_i is equal to zero; thus it has the same standard deviation as ξ_i.

After substituting Eq. (7) into the explicit forms of the element matrices for linear stiffness, incremental stiffness, mass, and aerodynamic force, and truncating the third and higher order terms, the stochastic element formulation including the effects of structural uncertainties and middle surface forces can be obtained in terms of random variables α_i ($i = 1, 2, \cdots, k$) in the following general form:

$$[c] = [c^{(0)}] + \sum_{r=1}^{k} [c_r^{(1)}]\alpha_r + \frac{1}{2}\sum_{r=1}^{k}\sum_{s=1}^{k} [c_{rs}^{(2)}]\alpha_r\,\alpha_s \tag{8}$$

where k is the total number of random variables in the element. The detailed deviation and the explicit forms for the element matrices can be found in Ref. 39.

The solution procedures for the stochastic vibration, buckling, and supersonic flutter analyses were proposed by the authors[28] to treat a flat laminated plate.

Results

To evaluate the present finite element formulation and solution procedure, to obtain results for quantifying the effects of the various uncertain parameters, and to gain physical insight into the subject problems, a series of vibration, buckling, and supersonic flutter analyses and related reliability studies for thin shells with structural uncertainties were performed. Interactive effects between the middle surface load and the aerodynamic pressure were also studied and quantified.

In all the examples, a sufficiently fine Gauss grid (5 × 5) was used for numerical integration to obtain the element linear stiffness, incremental stiffness, mass, and aerodynamic matrices. In all the examples, a portion of the shell with 1/4 wavelength in the circumferential direction and full length in the longitudinal direction was modeled and a 1 × 10 element mesh was used. This mesh was shown numerically to be sufficiently fine for all the present examples, i.e., this mesh yielded converged results as it was successively refined. All calculations were carried out using a CYBER 205 vectorized supercomputer at Purdue University.

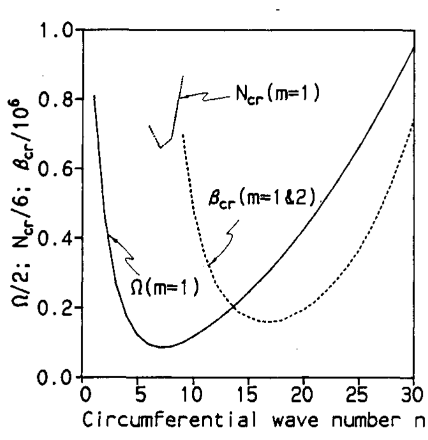

Fig. 3 Nondimensional natural frequency Ω, uniform axial buckling stress N_{cr}, and critical aerodynamic pressure β_{cr} of a simply supported laminated $(90/\pm45/0\ \text{deg})_s$ cylindrical shell.

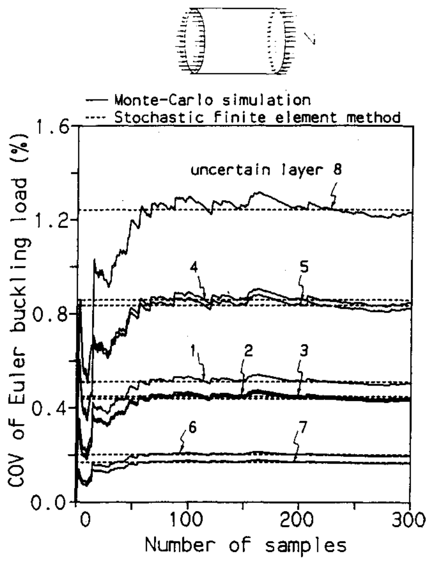

Fig. 4 The coefficients of variation of Euler buckling load of a simply supported graphite/epoxy laminated $(90/45/-45/0\ \text{deg})_s$ cylindrical shell with uncertain moduli of elasticity (only the numbered layer is uncertain and the remaining layers are not).

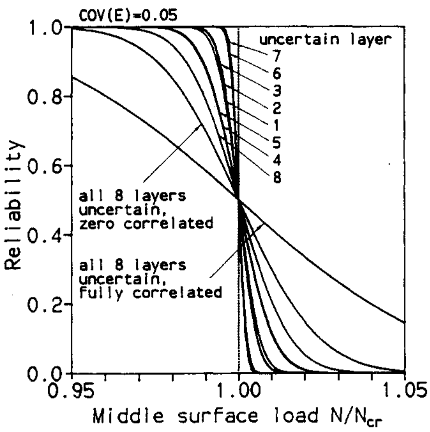

Fig. 5 Reliability of the simply supported graphite/epoxy laminated $(90/\pm45/0\ \text{deg})_s$ cylindrical shell under uniform compression with uncertain moduli of elasticity ($m = 1$, $n = 7$).

Case 1: Free Vibration, Buckling, and Supersonic Flutter of a Simply Supported Isotropic Cylindrical Shell

The example cylindrical shell studied was assumed as simply supported with length $L = 40$ in., radius $R = 20$ in., and thickness $h = 0.04$ in. The material properties were assumed as those for steel with $E = 30 \times 10^6$ psi, $\nu = 0.3$, and $\rho_s = 0.733 \times 10^{-3}$ lb-s²/in.⁴.

Figure 1 shows the results for the free vibration frequency ω, uniform axial buckling stress σ_{cr}, and critical aerodynamic pressure β_{cr} of the cylindrical shells for various circumferential wave number n. The frequency, buckling stress, and aerodynamic pressure are nondimensionalized, respectively, as

$$\Omega = \sqrt{\frac{\rho_s\,(1 - \nu^2)}{E}}\; R\,\omega \tag{9}$$

Table 1 Critical aerodynamic pressure β_{cr}, flutter frequency ω_f, and corresponding circumferential wave number n_{cr} for the example isotropic conical shells with fixed edges

Thickness $\times 10^{-3}$, in.	Circumferential wave number n_{cr}	Critical aerodynamic pressure β_{cr}, psi: This study	Critical aerodynamic pressure β_{cr}, psi: Ueda et al.[11]	Flutter frequency ω_f, Hz: This study	Flutter frequency ω_f, Hz: Ueda et al.[11]
1.181	22	17.09	17.15	1543	1510
1.575	21	30.68	30.72	1832	1800
1.969	19	48.50	50.01	1941	1910
2.362	18	70.62	72.16	2105	2080
2.756	18	96.82	99.31	2377	2350

Table 2 Nondimensional fundamental frequency and buckling load for the example simply supported laminated cylindrical shells

Stacking sequence, deg	Fundamental frequency $\omega L^2(\rho_s/E_T h^3)^{1/2}$: This study	Fundamental frequency: Jones and Morgan[42]	Buckling load $\sigma_{cr} L/E_T h^3$: This study	Buckling load: Jones and Morgan[42]
[0]	140.9(5)[a]	142.4	1462(7)[a]	1482
[0/90/0]	161.4(4)	164.7	1858(6)	1860
[0/90/0_8]	171.4(4)	171.1	1994(7)	1987
[0/90/0_{18}]	172.4(4)	174.1	2036(7)	2062
[0/90/0_{48}]	167.0(4)	170.7	1922(6)	1957

[a]The values in the parentheses are critical circumferential wave numbers.

$$N_{cr} = \frac{\sigma_{cr} R}{E h} \tag{10}$$

and

$$\beta_{cr} = \frac{\bar{\beta}_{cr} L^3}{D} \tag{11}$$

where D is the flexural rigidity of the shell.

The free vibration, buckling, and supersonic flutter characteristics of this cylindrical shell have been previously studied by Leissa,[40] Yamaki,[41] and Kobayashi,[7] respectively. Their results are also plotted in Fig. 1 for comparison. Good agreement is observed. It is interesting to see that the critical circumferential wave number for supersonic flutter is much higher (n_{cr} = 16) than that for free vibration and buckling (n_{cr} = 8).

Case 2: Supersonic Flutter of an Isotropic Conical Shell with Fixed Edges

The example conical shell studied has length L = 2.46 in., initial radius R = 1.57 in., and semivertex angle ψ = 14 deg. The material properties were assumed as $E = 18.29 \times 10^6$ psi, ν = 0.25, and $\rho_s = 0.744 \times 10^{-3}$ lb-s²/in.⁴.

Table 1 shows the results for critical aerodynamic pressure, flutter frequency, and each corresponding circumferential wave number for the conical shells with various thicknesses. The problem was studied previously by Ueda et al.[11] Their results are also shown in Table 1 for comparison. Good agreement is seen. It is seen for this example that when the thickness of the shell increases, the critical aerodynamic pressure and flutter frequency increase, and the critical circumferential wave number decreases correspondingly.

Case 3: Vibration and Buckling of Simply Supported Laminated Cylindrical Shells

The example cylindrical shells studied were assumed as simply supported with length L = 34.64 in., radius R = 10 in., and thickness h = 0.12 in. The material properties of each lamina were assumed as $E_L = 30 \times 10^6$ psi, $E_T = 0.75 \times 10^6$ psi, $G_{LT} = 0.375 \times 10^6$ psi, ν_{LT} = 0.25, and $\rho_s = 0.145 \times 10^{-3}$ lb-s²/in.⁴.

Table 2 shows the results for fundamental frequency, buckling load, and critical circumferential wave number of the cylindrical shells with various stacking sequences. This problem was studied previously by Jones and Morgan.[42] Their results are also shown in Table 2 for comparison. Good agreement is observed.

Case 4: Supersonic Flutter of a Simply Supported Orthotropic Conical Shell

The example conical shell studied was assumed as simply supported with length L = 61.37 in., initial radius R = 7.55 in., thickness h = 0.051 in., and semivertex angle ψ = 5 deg. The material properties were assumed as $E_L = 6.5 \times 10^6$ psi, ν_{LT} = 0.29, and $\rho_s = 0.833 \times 10^{-3}$ lb-s²/in.⁴.

Figure 2 shows the results for nondimensional critical aerodynamic pressure of the conical shells with various stiffness ratio (E_T/E_L). The problem was studied previously by Librescu.[12] His results are also plotted in Fig. 2 for comparison. Good agreement is seen.

Case 5: Free Vibration, Buckling, and Supersonic Flutter of a Simply Supported Laminated Cylindrical Shell

The example cylindrical shell studied was assumed as simply supported with length L = 40 in. and radius R = 20 in. The laminate construction was assumed as $[90/\pm 45/0 \text{ deg}]_s$ with thickness of each lamina equal to 0.005 in. The material properties of each lamina were assumed as $E_L = 20 \times 10^6$ psi, $E_T = 1.5 \times 10^6$ psi, $G_{LT} = 0.75 \times 10^6$ psi, ν_{LT} = 0.25, and $\rho_s = 0.145 \times 10^{-3}$ lb-s²/in.⁴.

Figure 3 shows the results for the nondimensional free vibration frequency, uniform axial buckling stress, and critical aerodynamic pressure of the cylindrical shells for various circumferential wave number n. Again, it is seen that the critical circumferential wave number for supersonic flutter is much higher (n_{cr} = 17) than that for free vibration and buckling (n_{cr} = 7).

Case 6: Buckling and Reliability of a Simply Supported Laminated Cylindrical Shell with Structural Uncertainties

The buckling and reliability of an eight-ply $(\theta_1/\theta_2/\theta_3/\theta_4)_s$ graphite/epoxy laminated cylindrical shell were investigated.

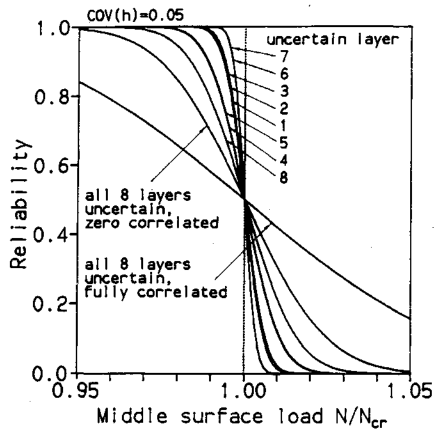

Fig. 6 Reliability of the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell under uniform compression with uncertain thickness (m = 1, n = 7).

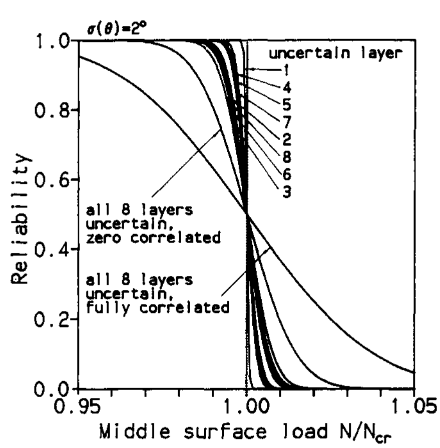

Fig. 7 Reliability of the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell under uniform compression with uncertain fiber orientations (m = 1, n = 7).

In this study, the variability of the volume fraction of fibers was accounted for using the modulus of elasticity as a random variable. The orientation of fibers is considered as a separate random variable. The other structural uncertainties include thickness of each lamina and initial geometric imperfections over the entire shell.

Physically speaking, not all the fibers in the same lamina have the same orientation. Practically speaking, it would be computationally very difficult to treat each of the fibers in the same lamina as a random variable. Thus the single random variable model in this study implicitly assumes that all fibers have the same orientation in the same lamina. Such a model is certainly more conservative and yields relatively lower reliability.

The cylindrical shell was assumed to be simply supported with length of 40 in. and radius of 20 in. The mean value and coefficient of variation for the thickness of each lamina were assumed as 0.005 in. and 5%, respectively. The material of each lamina was assumed as graphite/epoxy with $E_L = 20 \times 10^6$ psi, $E_T = 1.5 \times 10^6$ psi, $G_{LT} = 0.75 \times 10^6$ psi, and $\nu_{LT} = 0.25$. The coefficient of variation of the modulus of elasticity was assumed as 5% based on the tests for over 500 samples by Delmonte.[43] The mean values of fiber orientation θ_i (i = 1, . . . , 4) were assumed as 90, 45, −45, and 0 deg, respectively. The standard deviation of fiber orientation of lamina was assumed as 2 deg.

The shape of the initial geometric imperfection was assumed the same as the critical buckling mode with the following form:

$$w_i(x, \phi) = w_o \sin\frac{\pi x}{L} \sin 7\phi \qquad (12)$$

where x and ϕ are the cylindrical coordinates. The symbols m and n represent the number of longitudinal half-waves and the number of full circumferential waves, respectively. For this example, it was assumed that m and n equaled 1 and 7, respectively, to conform to the critical buckling mode. The random variable w_o was assumed as of Gaussian distribution with zero mean and standard deviation of $0.05h$.

The coefficients of variation of the Euler buckling load due to the uncertain modulus of elasticity, thickness, and fiber orientation in each lamina and initial imperfections of the entire shell were obtained using the stochastic finite element method proposed in this study. Since no alternative solutions were available to evaluate the present stochastic finite element solutions, a Monte-Carlo simulation method using 300 samples was employed. Figure 4 shows the results for coefficients of variation of buckling loads due to uncertain modulus of elasticity obtained by both methods. Good agreement is seen. It is seen that the uncertainty in the innermost (eighth) lamina has most significant effect on the variation of the buckling load in this case.

Based on the mean value of the buckling load and the standard deviation or coefficient of variation, Gaussian distribution curves were obtained for each of the uncertain structural parameters. Structural reliability for a given middle surface load and a specific uncertain parameter could be obtained

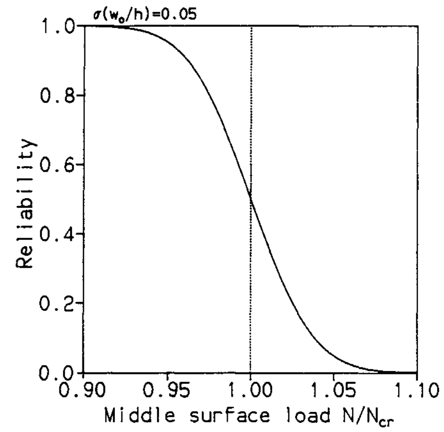

Fig. 8 Reliability of the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell under uniform compression with random imperfections (m = 1, n = 7).

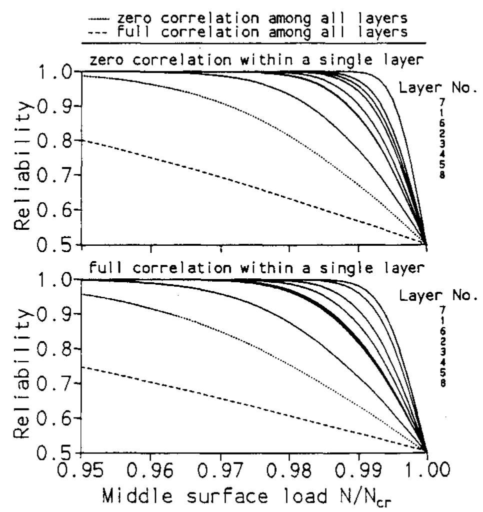

Fig. 9 Reliability of the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell under uniform compression with uncertain moduli of elasticity E and fiber orientations θ (m = 1, n = 7).

by integrating the portion of the area on the right-hand side of the given middle surface load and under the Gaussian distribution curve. The reliability curves thus obtained for uncertain modulus of elasticity, thickness, fiber orientation, and imperfection are plotted in Figs. 5, 6, 7, and 8, respectively.

In a previous study[28] of a simply supported eight-ply (90/±45/0 deg)$_s$ graphite/epoxy laminated thin square flat plate subjected to uniform in-plane compression, the reliability boundary curves for uncertain modulus of elasticity, thickness, and fiber orientation were found to become lower when each random parameter occurred in a lamina farther away from the middle surface. The degree of the individual effect of each lamina on the reduction of reliability was found to be symmetric about the middle surface of the flat plate. When the same flat plate was studied using supersonic flutter as a constraint, the reliability curves for the three respective uncertain parameters were found to be the lowest as each random parameter occurred in the second and seventh laminae (θ = 45 deg). Obviously, the reliability is not only dominated by the thickness location of the lamina relative to the neutral surface but also by other factors such as the fiber orientation of the lamina.

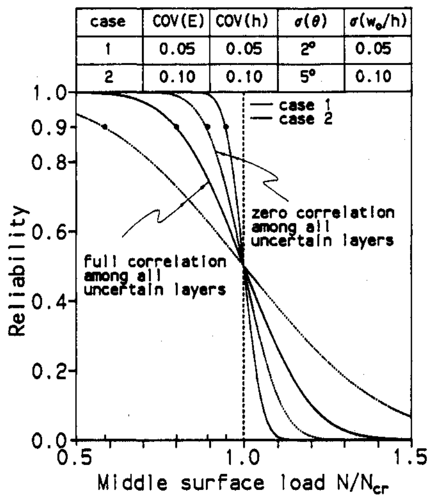

Fig. 10 Reliability of the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell under uniform compression with four uncertain parameters (m = 1, n = 7).

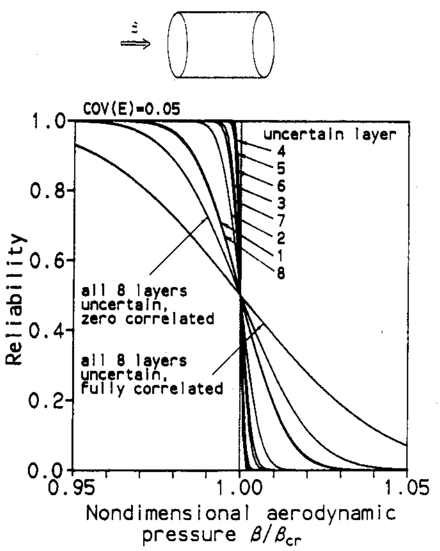

Fig. 11 Reliability of the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell in supersonic flow with uncertain moduli of elasticity (m = 1 and 2, n = 17).

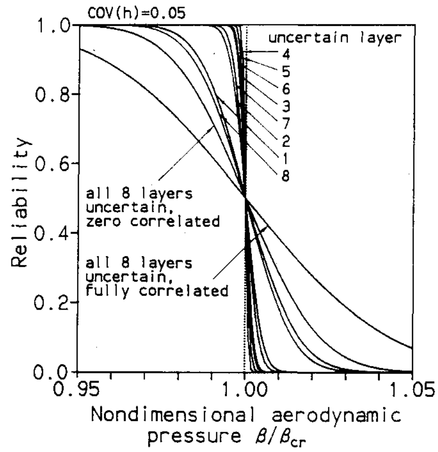

Fig. 12 Reliability of the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell in supersonic flow with uncertain thickness (m = 1 and 2, n = 17).

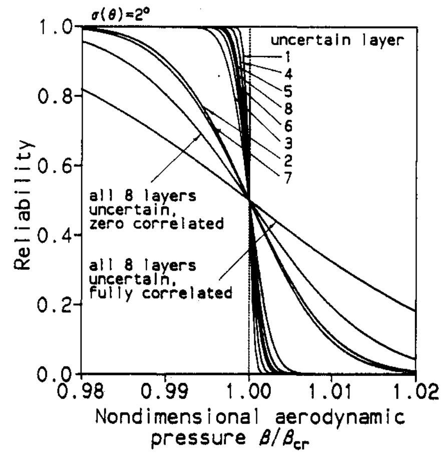

Fig. 13 Reliability of the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell in supersonic flow with uncertain fiber orientations (m = 1 and 2, n = 17).

When a flat plate is subjected to bending or vibration with small deflection, the neutral surface coincides with the middle surface. However, in the case of cylindrical shells, the membrane and bending behaviors are intricately coupled and the neutral surface no longer coincides with the middle surface even during the case of free vibration in the lowest mode.

It is seen in Figs. 5–7 that the individual structural uncertainty in each lamina has an effect on reducing the reliability boundary to a certain degree. Due to the complex coupling of membrane and bending behaviors, the location of each lamina along the thickness, and the fiber orientation of each lamina, the degree of the effect of each uncertain structural parameter in each lamina on the reduction of its reliability boundary curve due to buckling cannot be easily rank ordered by simple physical intuition or interpretation as in the case of plate. Such physical complexity further increases for the case of supersonic flutter. Figures 5–7 contribute by quantifying these reliability curves.

In Figs. 5–7, the reliability boundaries for the case with a random parameter existing in all eight layers but with zero correlation are shown to be lower than those with the same random parameter existing in any single lamina. The combined effects of eight laminae are certainly lower than that of any single uncertain lamina.

The reliability boundaries are the lowest when the random parameters in all eight layers are fully correlated. In the case of zero correlation among the eight laminae, the effect of uncertainty tends to compensate each other in a random fash-

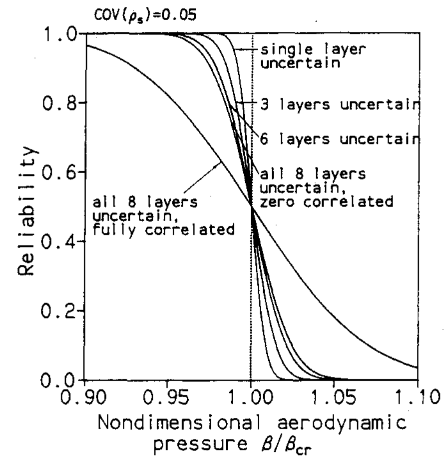

Fig. 14 Reliability of the simply supported graphite/epoxy laminated $(90/\pm45/0\ \mathrm{deg})_s$ cylindrical shell in supersonic flow with uncertain mass density of shell (m = 1 and 2, n = 17).

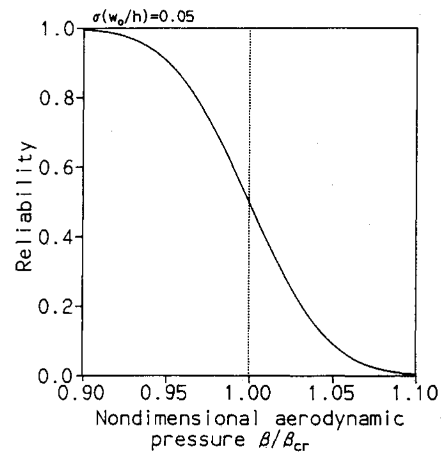

Fig. 15 Reliability of the simply supported graphite/epoxy laminated $(90/\pm45/0\ \mathrm{deg})_s$ cylindrical shell in supersonic flow with random imperfections (m = 1 and 2, n = 17).

ion. Thus higher reliability than the fully correlated case is observed.

Figure 8 shows that the random imperfection has an obvious effect on reducing the reliability boundary. When buckling is given as a failure criterion and for the values assumed in this study for COV(E), COV(h), $\sigma(\theta)$, and $\sigma(w_o/h)$, it can be seen that the effect of the random imperfection on reducing the reliability boundary is higher than those of the other three uncertain parameters in any single layer. When the uncertain parameters for all eight layers are fully correlated, the cases of COV(E) = 0.05 and COV(h) = 0.05 have similar reliability boundaries that are lower than the case of $\sigma(\theta)$ = 2 deg and the case of $\sigma(w_o/h)$ = 0.05. It is hard to evaluate the relative importance of these four random uncertainties as these effects are dependent on the specific values assumed.

One must quantify the practical ranges of values of these random variables as based on the actual fabrication process. Reversely, one could also, based on the method and results presented here, give limits of these random variables to control the fabrication process so that certain buckling strength can be achieved.

It is of interest to assume only two random variables, E and θ, as statistically independent within a single lamina and to study the effect of this correlation on the reliability due to buckling. Figure 9 shows the combined effects of two random parameters, E and θ, on the structural reliability boundaries. Two extreme correlated conditions, zero and full correlations, within each single layer were considered. It is of interest to see that effects of uncertainties in each layer on the reduction of its reliability boundary are similar for both correlated conditions. The zero correlated case results in 10 slightly higher reliability curves than the fully correlated case. This seems to be due to the compensation effect of these two uncertainties in each single layer. The 10 curves in each case are rank ordered based on their reliability. The orders are the same for both cases. However, these orders are not the same as those for the cases with only one uncertainty, i.e., E for Fig. 5 and θ for Fig. 7.

In practical application, it is possible that all the above uncertain parameters occur simultaneously in certain random fashion. Figure 10 shows the combined effects of all these uncertain parameters on the structural reliability boundaries for the subject shell under middle surface compression. Two sets of coefficients of variation (or standard deviations) of the structural uncertainties were assumed, as listed in Fig. 10, to cover ranges of practical significance. Two extreme conditions, zero and full correlations, among these structural uncertainties were assumed. It is of interest to observe these curves by choosing a given value of structural reliability, say 0.90. For this reliability value, the allowable nondimensional middle surface compression decreases to 0.95 and 0.80 for the zero and fully correlative conditions, respectively, for case 1; and to 0.89 and 0.58, respectively, for case 2. Significant reductions of the allowable middle surface load of the uncertain shell are seen.

Case 7: Supersonic Flutter and Reliability of a Simply Supported Laminated Cylindrical Shell with Structural Uncertainties

It is of interest to study the reliabilities of the present shell, same as that of the previous example, considering supersonic

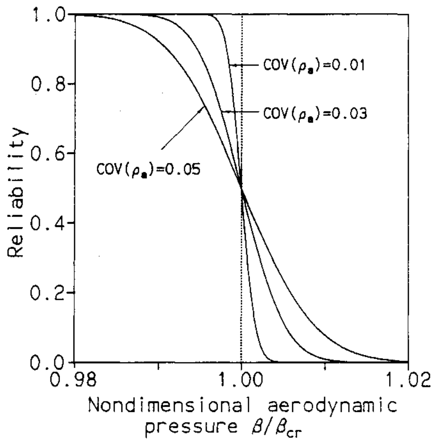

Fig. 16 Reliability of the simply supported graphite/epoxy laminated $(90/\pm45/0\ \mathrm{deg})_s$ cylindrical shell in supersonic flow with uncertain mass density of air (m = 1 and 2, n = 17).

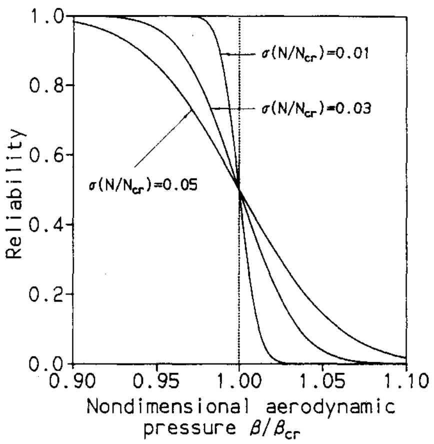

Fig. 17 Reliability of the simply supported graphite/epoxy laminated $(90/\pm45/0\ \mathrm{deg})_s$ cylindrical shell in supersonic flow under uniformly distributed random middle surface loads (m = 1 and 2, n = 17).

flutter as a failure criterion. The same uncertain structural parameters were considered but with the addition of a mass density. The mean value and coefficient of variation for the mass density of each lamina were assumed as 0.145×10^{-3} lb-s²/in.⁴ and 5%, respectively. The aerodynamic pressure due to supersonic potential flow was described by the two-dimensional quasisteady supersonic theory. To study the effect of randomness of airflow on the determination of flutter characteristic, the mass density of air was also considered as an additional uncertain parameter.

In this example, the reliability criterion was assumed as the probability of a given aerodynamic pressure to be less than the critical aerodynamic pressure. The reliability curves for the shell due to uncertain modulus of elasticity [COV(E) = 0.05], thickness [COV(h) = 0.05], fiber orientation [$\sigma(\theta)$ = 2 deg], mass density of shell [COV(ρ_s) = 0.01, 0.03, and 0.05, respectively], geometric imperfection [$\sigma(w_o/h)$ = 0.05], and mass density of air [COV(ρ_a) = 0.01, 0.03, and 0.05, respectively] were obtained and plotted in Figs. 11–16, respectively.

It is seen in Figs. 11–13 that each structural uncertainty in each layer has an effect on reducing the reliability boundary to a certain degree. As explained in the previous section in

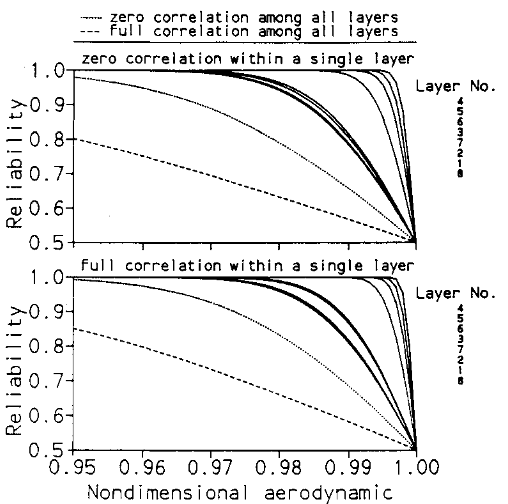

Fig. 18 Reliability of the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell in supersonic flow with uncertain moduli of elasticity E and fiber orientations θ (m = 1 and 2, n = 17).

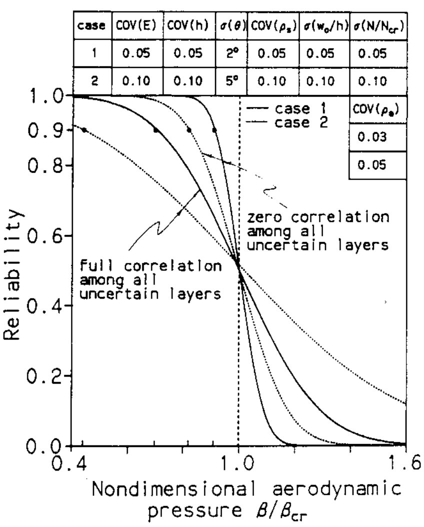

Fig. 19 Reliability of the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell in supersonic flow with seven uncertain parameters (m = 1 and 2, n = 17).

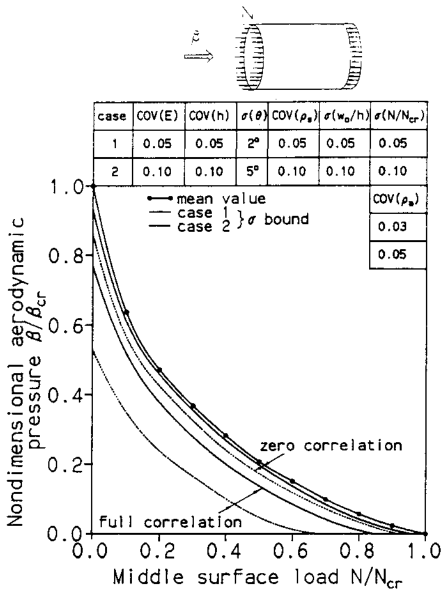

Fig. 20 Interactive effects between uniform compression and aerodynamic pressure for the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell with seven uncertain parameters (m = 1 and 2, n = 17).

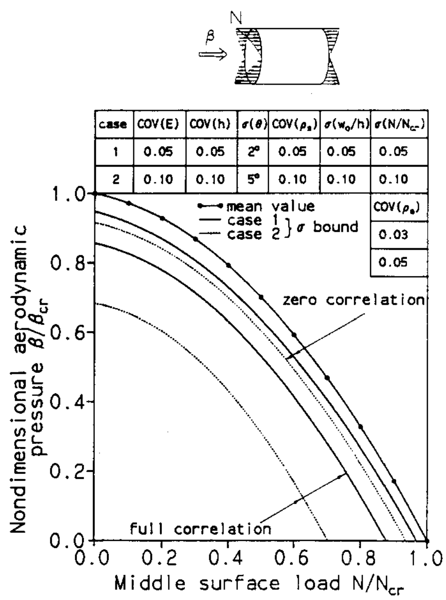

Fig. 21 Interactive effects between linearly distributed middle surface load and aerodynamic pressure for the simply supported graphite/epoxy laminated (90/±45/0 deg)$_s$ cylindrical shell with seven uncertain parameters (m = 1 and 2, n = 17).

conjunction with the case of buckling, due to factors such as the coupling of the membrane and bending behaviors, the location of the neutral surface, the location of each lamina along the thickness, and the fiber orientation of each lamina, etc., the degree of influence of each uncertain structural parameter of each lamina on the reduction of its reliability boundary curve cannot be easily rank ordered by simple physical intuition or interpretation as in the case of flat plate. Such physical intuition and insight become even more complex for the case of flutter. Figures 11–13 contribute by quantifying these reliability boundaries.

In Figs. 11–13, the reliability boundary curves for the case with a random parameter existing in all eight layers but with zero correlation are shown to be lower than any of those for just one uncertain layer. The reliability boundary curves are the lowest when the random parameters in all eight layers are fully correlated.

Figures 14–16 show the obvious effects of the three respective uncertain parameters, i.e., the mass density of single and all layers of the shell, random imperfection of the shell, and mass density of the air, on the reliability boundaries. The three figures provide quantifications of the results for this case as well as some physical insight.

For more practical application, the effect of random but uniformly distributed middle surface load was also considered. The mean value for the uniformly distributed random middle surface load could be assumed as having any value between zero and the Euler buckling load (N_{cr}). As will be seen in later figures, 11 different mean values (N/N_{cr} = 0, 0.1, 0.2, · · · , 0.9, and 1) were assumed when the interactive effects between the middle surface load and aerodynamic pressure were calculated. As a first step, the mean value was assumed as zero. Figure 17 shows the reliability curves for the shell under uniformly distributed zero-mean random middle surface loads with different standard deviations, $\sigma(N/N_{cr})$ = 0.01, 0.03, and 0.05.

When supersonic flutter is given as a failure criterion and for the values assumed in this study for COV(E) = 0.05, COV(h) = 0.05, $\sigma(\theta)$ = 2 deg, $\sigma(w_o/h)$ = 0.05, COV(ρ_s) = 0.05, COV(ρ_a) = 0.05, and COV(N/N_{cr}) = 0.05, it can be seen that the effect of the random middle surface loads on reducing the reliability boundary is higher than those of the other six uncertain parameters in any single layer. When the uncertain parameters for all eight layers are fully correlated, the reliability boundary for the case of COV(ρ_s) = 0.05 is lower than those for the other six uncertain parameters. It is hard to evaluate the relative importance these seven random uncertainties as these effects are dependent on the specific values assumed.

It is of interest to assume only two random variables, E and θ, as statistically independent within a single lamina and to study the effect of this correlation on the reliability due to supersonic flutter. Figure 18 shows the combined effects of two random parameters, E and θ, on the structural reliability boundaries. Two extreme correlated conditions, zero and full correlations, within each single layer were considered. It is of interest to see that effects of uncertainties in each layer on the reduction of its reliability boundary are similar for both correlated conditions. The zero correlated case results in 10 slightly higher reliability curves than the fully correlated case. This seems to be due to the compensation effect of these two uncertainties in each single layer. The 10 curves in each case are rank ordered based on their reliability. The orders are the same for both cases. Furthermore, the orders are identical to those for the case of uncertain modulus of elasticity, since the uncertain modulus of elasticity has much more pronounced effect on reducing the reliability boundary than fiber orientation in the same layer, as shown in Figs. 11 and 13.

Figure 19 shows the combined effects of all these uncertain parameters on the structural reliability of the shell in supersonic flow. Two sets of coefficients of variation (or standard deviations) for these uncertain parameters were assumed, as listed in Fig. 19. Two extreme conditions, zero and full correlations, among these seven uncertain parameters were also assumed. It is interesting to see these curves by choosing a given value of structural reliability, say 0.90. For this reliability value, the allowable nondimensional aerodynamic pressure decreases to 0.91 and 0.70 for the zero and fully correlative conditions, respectively, for case 1, and to 0.82 and 0.44, respectively, for case 2. Significant reductions of the allowable aerodynamic pressure for this uncertain shell are observed.

Case 8: Supersonic Flutter of a Simply Supported Laminated Cylindrical Shell with Structural Uncertainties Under Uniform and Linear, Nonzero-Mean, Random Middle Surface Loads

To study the interactive effects between the middle surface load and aerodynamic pressure, the same shell was further studied by including the random effect of nonzero-mean middle surface loads. The mean values of the middle surface loads were varied from zero to its Euler buckling load N_{cr} by 10 equal increments. Two sets of coefficients of variation (or standard deviations) for all the seven uncertain parameters were assumed, as listed in Fig. 20. Both zero and full correlative conditions among all the uncertain parameters were considered.

Figure 20 shows the mean values and the standard deviations of the critical aerodynamic pressure of the shell with the two sets of uncertain parameter values under uniformly distributed middle surface loads in the form of interactive stability boundaries. The effects of these random parameters with currently assumed values on the reduction of these stability boundaries were quantified in Fig. 20. It is apparent that for case 2 with full correlation of random parameters, this boundary is significantly reduced. It is of interest to note that the relationship between the critical aerodynamic pressure and the uniformly distributed middle surface load is obviously nonlinear.

The random effect of linearly distributed middle surface load was also considered. Figure 21 shows the mean values and standard deviations of the critical aerodynamic pressure of the shell with the two sets of uncertain parameter values under linearly distributed random middle surface loads. Again, the effects of the currently assumed random parameters on the reduction of the stability boundaries were quantified. It is seen that the relationship between the critical aerodynamic pressure and the linearly distributed middle surface load is also nonlinear, but with convexity opposite to that for the uniformly distributed middle surface load (Fig. 20).

Concluding Remarks

A stochastic 48-DOF quadrilateral laminated thin shell element with probabilistic modulus of elasticity, mass density, thickness, fiber orientation, geometric imperfection, middle surface load, and mass density of the air has been formulated. The solution procedure was based on the mean-centered, second-moment perturbation technique. The effects of these uncertain parameters on the vibration, buckling, supersonic flutter, and reliability have been studied.

The present results have been compared with alternative solutions in the case of deterministic analyses (Tables 1 and 2, and Figs. 1 and 2). Monte-Carlo simulation method has been used to verify a set of coefficients of variation for Euler buckling loads (Fig. 4).

The numerical results indicate that the individual structural uncertainty in each lamina has an effect on reducing the reliability boundary of shells to a certain degree in the case of buckling. Due to the complex coupling of membrane and bending behaviors, the location of each lamina along the thickness, and the fiber orientation of each lamina, the degree of the effect of each uncertain structural parameter in each lamina on the reduction of its reliability boundary curve due to buckling cannot be easily rank ordered by simple physical intuition or interpretation as in the case of plate. Such physical complexity further increases for the case of supersonic flutter. When the uncertain parameters were assumed to exist simultaneously in all eight layers, it is observed that the zero correlated cases result in higher reliabilities than the fully correlated cases due to the compensation effects among the layers in a random fashion.

In this study, ranges of values for each random parameter were assumed, and the results obtained quantify the effects of these parameters on the reliabilities of the present laminated shells subjected to middle surface force and supersonic aerodynamic pressure.

The interactive effects between the middle surface load and aerodynamic pressure were also studied and quantified for the present sets of assumed random parameters. Such interactive stability boundaries were shown to be nonlinear when the middle surface loads were both uniformly and linearly distributed.

These quantified data may provide structural designers with some physical insight into the possible individual and combined effects of these uncertain parameters. Such physical insight and the quantified data may be helpful in the design of more reliable laminated composite shell structures.

The present formulation and solution procedure are general and simple so that extensions can be made to formulate any other types of laminated thin shell finite element in a stochastic fashion.

It is noted that in addition to random error, random variables could also be subject to systematic bias. Since the randomness considered in this paper is formulated through a perturbation scheme retaining terms up to second order, the effect of the systematic bias is neglected. Such an effect can be incorporated in the analysis when higher-order terms in the perturbation expansion are kept. Such a higher-order reliability analysis would provide more interesting and accurate results and is a worthy topic for extension of this study.

Extensions can also be made to include some more interesting effects, such as transverse shear deformations for thicker laminated shells, structural and aerodynamic dampings, as well as structural and aerodynamic nonlinearities, etc.

Acknowledgments

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